$\tau$-Liveness Enforcement in Petri Nets
Based on Structural Net Properties

Marian V. Iordache and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
iordache.1, antsaklis.1@nd.edu
Introduction

Given a PN $\mathcal{N} = (P, T, F, W)$ and $\mathcal{T} \subseteq T$:

- $(\mathcal{N}, \mu_0)$ is $\mathcal{T}$-live if all transitions in $\mathcal{T}$ are live.
- $\mathcal{N}$ can be made $\mathcal{T}$-live (or $\mathcal{T}$-liveness is enforceable in $\mathcal{N}$) if
  $\exists \mu_0 \exists$ supervisor $\Xi$ such that $(\mathcal{N}, \mu_0, \Xi)$ is $\mathcal{T}$-live.
- Liveness is $\mathcal{T}$-liveness for $\mathcal{T} = T$.
What are the initial markings for which a PN can be made $T$-live?

\[ [C_1 :] \quad \mu_1 + \mu_3 \geq 1 \] \hspace{1cm} (1)

\[ [C_2 :] \quad \mu_2 + \mu_3 \geq 1 \] \hspace{1cm} (2)

\[ [C_3 :] \quad \mu_1 + \mu_2 + \mu_3 \geq 2 \] \hspace{1cm} (3)

The control place (monitor) $C_3$ is useless.

- Let $L\mu \geq b$ describe (1) and (2).
- Let $L_0 \mu \geq b_0$ describe (3).

The PN is live for all initial markings $\mu_0$ satisfying

\[ L\mu_0 \geq b \text{ and } L_0 \mu_0 \geq b_0 \] \hspace{1cm} (4)

when supervised according to the constraint $L\mu \geq b$. 
Defining the $\mathcal{T}$-liveness enforcing procedure

Given an arbitrary PN and $\mathcal{T}$, the procedure finds matrices $L, L_0, b, b_0$, such that the PN is $\mathcal{T}$-live for all initial markings $\mu_0$ satisfying

$$L\mu_0 \geq b \quad \text{and} \quad L_0\mu_0 \geq b_0 \tag{5}$$

when supervised according to the constraint $L\mu \geq b$. 
Let $\mathcal{N} = (P, T, F, W)$ be a PN.

We call $\mathcal{N}$ *PT-ordinary* if

$$\forall (p, t) \in F: W(p, t) = 1$$

$\mathcal{N}$ has *asymmetric choice* if

$$\forall p_1, p_2 \in P: p_1 \cap p_2 \neq \emptyset \Rightarrow p_1 \subseteq p_2 \lor p_2 \subseteq p_1$$
An **active subnet** is a PN subnet which can be made live. Formally:

Given \( \mathcal{N} = (P, T, F, W) \) of incidence matrix \( D \), \( \mathcal{N}^A = (P^A, T^A, F^A, W^A) \) is an **active subnet** of \( \mathcal{N} \) if there is \( x \geq 0, x \neq 0 \), such that \( Dx \geq 0 \) and \( T^A = ||x|| \), \( P^A = T^A \circ, F^A = F \cap \{ (T^A \times P^A) \times (P^A \times T^A) \} \) and \( W^A \) is \( W \) restricted to \( F^A \).

If \( \mathcal{T} \subseteq T^A \) and there is no active subnet \( \mathcal{N}_1^A = (P_1^A, T_1^A, F_1^A, W_1^A) \) such that \( \mathcal{T} \subseteq T_1^A \) and \( T_1^A \subseteq T^A \), we say that \( \mathcal{N}^A \) is a \( \mathcal{T} \)-minimal **active subnet** of \( \mathcal{N} \).
A **siphon** is a set of places $S \neq \emptyset$ such that $\bullet S \subseteq S \bullet$.

$S$ is an **active siphon** with respect to an active subnet, if it is a siphon which includes one or more places of that subnet.

$S$ is a **minimal active siphon**, if there is no other siphon $S' \subseteq S$ active w.r.t. the same active subnet.

The only nonempty active subnet has $T^A = \{t_1, t_2, t_3\}$.

The active siphons are $\{p_1, p_3\}$, $\{p_2, p_3, p_4\}$ and $\{p_1, p_2, p_3, p_4\}$; the first two are also minimal.

A siphon $S$ is **controlled** w.r.t. a set of PN initial markings if for all reachable markings the total marking of $S$ is nonzero.
**Theoretical Foundation**

**Theorem.** Given a PT-ordinary asymmetric-choice net $\mathcal{N}$, let $\mathcal{N}^A$ be a $\mathcal{T}$-minimal active subnet. If all minimal active siphons w.r.t. $\mathcal{N}^A$ are controlled, the PN is $\mathcal{T}$-live.

The PN is $\mathcal{T}$-live for $\mathcal{T} = \{t_1, t_2, t_3\}$.

There is a single $\mathcal{T}$-minimal active subnet $\mathcal{N}^A$ (the one with $\mathcal{T}^A = \mathcal{T}$.)

All minimal active siphons w.r.t. $\mathcal{N}^A$ are controlled: $\{p_1, p_3\}$, $\{p_1, p_4\}$, $\{p_2, p_3, p_6\}$, and $\{p_2, p_5, p_6\}$.
$\mathcal{T}$-liveness supervisors are generated by iteratively correcting deadlock situations. This involves the following:

1. siphon control
2. transformations to PT-ordinary and asymmetric choice Petri nets
3. active subnet computation
**Procedure**

**Siphon Control**

**Siphon control:** at every iteration, all uncontrolled minimal active siphons $S$ are controlled by enforcing:

$$\sum_{p \in S} \mu(p) \geq 1$$

(6)

Depending on the structural properties, (6) can be enforced by adding a control place (monitor) to the PN or by only requiring the initial marking to satisfy (6).

$C_1$ controls $\{p_1, p_3\}$ and $C_2$ controls $\{p_2, p_3\}$.

$\{C_1, p_2\}$ and $\{C_2, p_1\}$ controlled by requiring $\mu_0(C_1) + \mu_0(p_2) \geq 1$ and $\mu_0(C_2) + \mu_0(p_1) \geq 1$
Procedure

**Transformation to PT-ordinary PNs**

In the example, any inequalities on the original PN are changed as follows:

\[
\begin{align*}
\mu(p_1) &\rightarrow \mu(p_1) \\
\mu(p_2) &\rightarrow \mu(p_2) + \mu(p_{1,1}) \\
\mu(p_3) &\rightarrow \mu(p_3) + \mu(p_{1,2}) + 2\mu(p_{1,1})
\end{align*}
\]

**Transformation to AC nets**

In the example, any inequalities on the original PN are changed as follows:

\[
\begin{align*}
\mu(p_1) &\rightarrow \mu(p_1) + \mu(p_3) \\
\mu(p_2) &\rightarrow \mu(p_2)
\end{align*}
\]

In general: \(\mu(p_i) \rightarrow \mu(p_i) + \sum_j k_j \mu(p_{i,j})\)
The computation of a $\mathcal{T}$-minimal active subnet reduces to:

Find $x \geq 0$, $x_i > 0 \ \forall t_i \in \mathcal{T}$, such that $Dx \geq 0$ and there is no other $y \geq 0$, $Dy \geq 0$, $y_i > 0 \ \forall t_i \in \mathcal{T}$, such that $\|y\| \subset \|x\|$.

At every iteration the active subnet is *updated* by repeating the changes done to the PN in the active subnet.
Procedure

Input: The target PN $\mathcal{N}_0$ and the set $\mathcal{T}$
Output: Two sets of constraints $(L, b)$ and $(L_0, b_0)$

repeat

1. Transform the current net to a PT-ordinary AC PN.

2. Compute the $\mathcal{T}$-minimal active subnet.

3. For every uncontrolled minimal active siphon $S$ do
   If $S$ needs to be controlled with a control place then
     add control place to Petri net and inequality in $(L, b)$.
   Else
     add inequality to $(L_0, b_0)$.

until no uncontrolled minimal siphon is found at 2.

Restrict the constraints $(L, b)$ and $(L_0, b_0)$ to the places of $\mathcal{N}_0$.

$\mathcal{T}$-liveness is enforced for all initial markings $\mu_0$ such that

$$L\mu_0 \geq b \text{ and } L_0\mu_0 \geq b_0$$

by supervising $\mathcal{N}_0$ according to $L\mu \geq b$. 
Theoretical Results

**Theorem.** The supervisors generated by the $\mathcal{T}$-liveness procedure enforce $\mathcal{T}$-liveness.

**Theorem.** Given a PN and $\mathcal{T}$, if the PN has a single $\mathcal{T}$-minimal active subnet and the procedure terminates, the generated supervisor is least restrictive.

A supervisor generated by the procedure is said to be least restrictive when:

- The set of initial markings $\mu_0$ for which liveness is enforcible is

$$L\mu_0 \geq b \land L_0\mu_0 \geq b_0$$  \hspace{1cm} (7)

- For all initial markings $\mu_0$ satisfying (7), there is no $\mathcal{T}$-liveness enforcing supervisor less restrictive.
\( T \)-Liveness Enforcement Example

\[
L = [2, 2, 1], \ b = 2, \ L_0 = [] \text{ and } b_0 = []
\]
**Performance**

- The procedure makes no assumption on the PN structure; it is applicable to PNs which may be unbounded and generalized. Furthermore, it can be extended to PNs with uncontrollable and unobservable transitions.

- The procedure does not assume a given initial marking, but rather provides the constraints that the initial markings must satisfy for the supervisor to be effective.

- If the procedure terminates and the PN has a single $\mathcal{T}$-minimal active subnet, the procedure provides the least restrictive $\mathcal{T}$-liveness enforcing supervisor.

- When the procedure is used for liveness enforcement, the whole net is the single $\mathcal{T}$-minimal active subnet. Therefore, the supervisors generated by the procedure in this case are least restrictive.

  - Procedure termination is not guaranteed.

  - The procedure will not terminate for any PN with a single $\mathcal{T}$-minimal active subnet and with the property that the set of markings for which $\mathcal{T}$-liveness can be enforced is not the set of integer points of a polyhedron.
Performance

- The procedure may perform in each iteration computationally expensive operations (checking whether a siphon is uncontrolled may involve solving integer programs; finding the minimal siphons of a PN may also be computationally complex).

+ All computations are performed off-line. Very little computation is required to run a supervisor on-line.

+ The procedure allows fully automated computer implementation (and we have implemented it).